Formulae for approximating the elastic properties of a composite member composed of members carrying a force in parallel and series [R01]

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1 Abstract

This calculation supports a model of the stress and strain distribution in a Tokamak toroidal field (TF) coil. The model assumes that the TF coil structure is composed of uniform, long, axisymmetric layers of transverse-isotropic materials. However, in a true TF coil the winding pack layer is composite, composed of many individual windings, each with their own elastic properties. This document derives formulae for approximating the effective, smeared elastic properties of the composite material. We approximate the axial Young's modulus and the axial-transverse Poisson's ratio of two members carrying an axial force in parallel and in series. These quantities are sufficient to approximate the axial and transverse Young's modulus and the transverse and axial-transverse Poisson's ratio of the TF coil winding pack, given certain assumptions about the winding pack layout. The findings can be simply formulated as weighted averages.

2 Introduction

Figure 1 summarizes the assumed geometry of the two cases considered: a) Two members carrying a force in parallel and b) two members carrying a force in series. The geometry of the problem is such that the force is applied in direction $\hat{j}$. The members have Young's modulus in the $\hat{j}$ direction E_j. The members are assumed to be transverse-isotropic, such that the $j, \perp$ Poisson's ratio can be written $\nu_{j,\perp} = \nu_{j,i} = \nu_{j,k}$.

As the order of indices of the Poisson's ratio may not be uniform in the literature, here we

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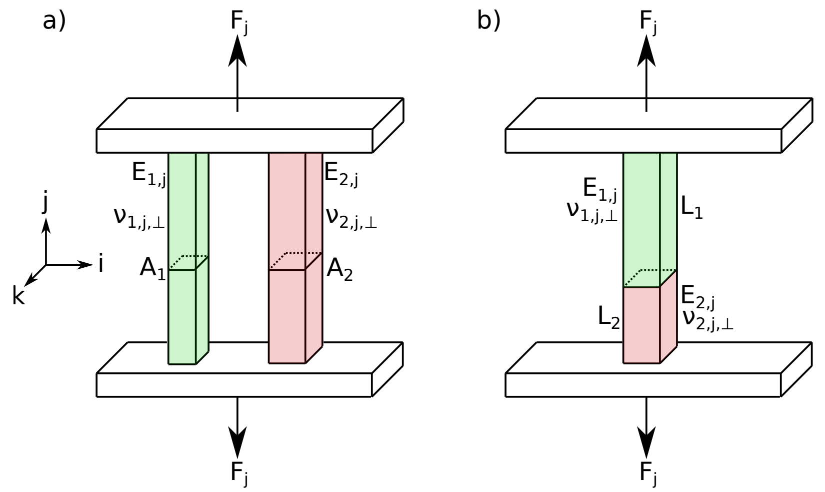

Figure 1: The assumed geometry of our problems: a) Members 1 and 2 carrying force F in parallel. b) Members 1 and 2 carrying force F in series. F is in the $\hat{j}$ direction. E_j is the Young's modulus in the $\hat{j}$ direction. $\nu_{j,\perp} = \nu_{j,i} = \nu_{j,k}$ is the Poisson's ratio in the j-transverse direction, assumed to be equal in the two transverse directions $\hat{i}, \hat{k}$.

clarify that we are using the convention

$$\nu_{j,\perp} = \left(\frac{\partial \epsilon_\perp}{\partial \epsilon_j}\right)_{\sigma_j} \tag{1}$$

where ϵ is the strain and $(\cdot)_{\sigma_j}$ indicates that this is the deformation under stress in the $\hat{j}$ direction only.

This report contains the following sections:

- Section 3: A simple description of the findings of this calculation
- Section 4: A summary of the assumptions involved in this calculation
- Section 5: The derivation of the compositing formulae
- Section 6: The application of these formulae to the PROCESS TF coil model

This work is in support of the PROCESS 0D systems code.[1, 2, 3, 4] As of this writing, there are a few places in the TF coil and Central Solenoid (CS) coil stress computation where bespoke analytic results based on isotropic forms of these formulae are used.[5] This work reproduces these results, extends them to anisotropy, and introduces composition operations which makes it easier to composite many (> 2) members.

The smeared elastic properties are passed to an axisymmetric extended plane strain[6] solver, which is responsible for determining the radial profiles of the stresses and strains within the TF coils.[7, 8]

3 Simple statement of the findings

We find that: the smeared axial Young's modulus of two members carrying an axial force in parallel is the cross-sectional-area-weighted average of the individual Young's moduli.

$$E_{j,par} = \frac{\sum E_j A}{\sum A} = \langle E_j \rangle_A \tag{2}$$

The smeared axial-transverse Poisson's ratio of these two members is the cross-sectional-area-weighted average of the individual axial-transverse Poisson's ratios.

$$\nu_{j,\perp,par} = \frac{\sum \nu_{j,\perp} A}{\sum A} = \langle \nu_{j,\perp} \rangle_A \tag{3}$$

The smeared axial Young's modulus of two members carrying an axial force in series is the inverse of the length-weighted average of the inverse of the individual Young's moduli.

$$E_{j,ser} = \frac{\sum L}{\sum L/E_j} = \langle E_j^{-1} \rangle_l^{-1} \tag{4}$$

The smeared axial-transverse Poisson's ratio of these two members is the weighted average of the individual axial-transverse Poisson's ratios, weighted by the ratios of length to axial Young's modulus.

$$\nu_{j,\perp,ser} = \frac{\sum \nu_{j,\perp} L/E_j}{\sum L/E_j} = \langle \nu_{j,\perp} \rangle_{L/E_j} \tag{5}$$

4 Summary of the assumptions

4.1 Axial Young's modulus of two members carrying an axial force in parallel

The central assumption of this computation is that the total axial strain ϵ_j is uniform and equal between the two members, $\epsilon_{1,j} = \epsilon_{2,j}$.

This assumption is valid when the boundary conditions enforce uniform axial deformation, or when the members are long enough that shear between the members enforces this condition. When the members being composited are themselves made of series-composited sub-members, the effect of neighboring parallel members whose stiffness may vary differently along their length is not considered. See Figure 2 for an illustration of this shortcoming.

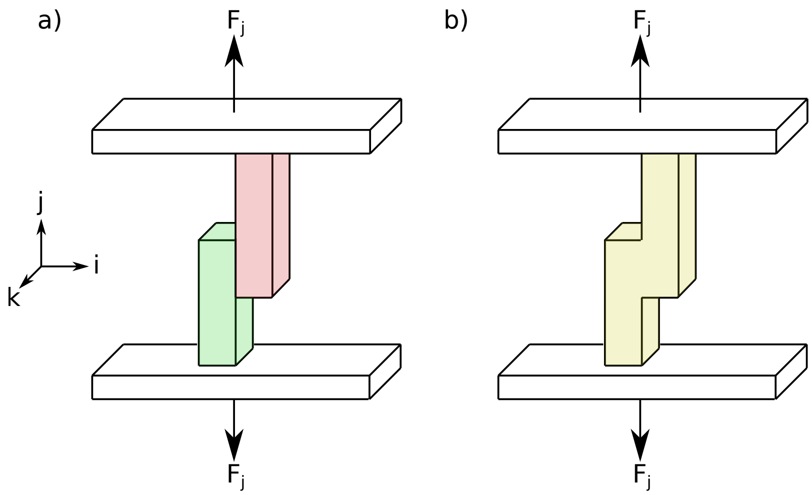

Figure 2: An example of an effect which is not captured by this model: a) The E_j of the green and red members are zero, as they are each serial-composited with an empty space. Therefore the E_j of the parallel-composited member is zero. However, in b) it is revealed that these members are joined, and so constrain each other's axial deformation via shear. $E_j \neq 0$ in this case. This behavior is not captured by the model.

4.2 Axial-transverse Poisson's ratio of two members carrying an axial force in parallel

The central assumptions of this computation are: that the dynamics transverse to j are isotropic, and that transverse expansion/contraction of one member as a result of axial stress is unimpeded by the presence of the other members.

This assumption is valid when the axial-transverse Poisson's ratios of the two members are similar, or when relative transverse motion between the members is permitted due to a frictionless interlayer, or when the members are arranged in a transverse tessellation which permits differential expansion/contraction. The effect of neighboring parallel members constraining the transverse expansion/contraction is not considered. See Figure 3 for an illustration of this shortcoming.

4.3 Axial Young's modulus of two members carrying an axial force in series

The central assumption of this computation is that the total axial stress σ_j is equal between the two members, $\sigma_{1,j} = \sigma_{2,j}$.

This assumption is valid when the two members being composited are uniform, and are free to axially deform by differing amounts. The effect of neighboring parallel members constraining the

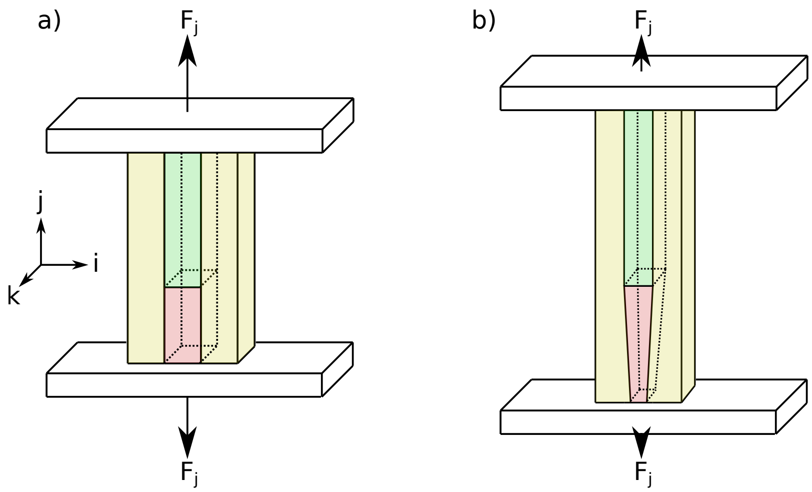

Figure 3: An example of another effect which is not captured by this model. Suppose the red member has a much higher Poisson's ratio than the green or yellow members. a) shows the configuration before force is applied. b) shows the configuration after force is applied. Independent of the green and yellow members, the red member would contract transversely under Poisson effects much more than shown. It is unable to contract by this amount due to the constraining presence of the green and yellow members. This behavior is not captured by the model.

axial deformation is not considered. See Figure 2 for an illustration of this shortcoming.

4.4 Axial-transverse Poisson's ratio of two members carrying an axial force in series

The central assumptions of this computation are again: That the dynamics transverse to j are isotropic, and that transverse areal contraction of one member as a result of axial stress is unimpeded by the presence of the other members.

This assumption is valid when the axial-transverse Poisson's ratios of the two members are similar, or when the members are arranged in a transverse-axial tessellation which permits differential expansion/contraction. The effect of neighboring parallel members constraining the transverse deformation is not considered. See Figure 3 for an illustration of this shortcoming.

5 Derivation of the compositing formulae

5.1 Axial Young's modulus of two members carrying an axial force in parallel

For the geometry of this case see Figure 1 (a). Take $E_{1,j}$ to be the Young's modulus in the j direction of Member 1, and $E_{2,j}$ to be the Young's modulus in the j direction of Member 2. Take A_1 to be the cross-sectional area in the j direction of Member 1, and A_2 to be the cross-sectional area in the j direction of Member 2.

Consider $E_{3,j}$ to be the effective Young's modulus in the j direction that a member of cross-sectional area $A_3 = A_1 + A_2$ would have to have, in order to produce the same strain ϵ_j under the same force $F_j = \sigma_{1,j} A_1 + \sigma_{2,j} A_2$ as the two individual members in parallel. $\sigma_{1,j}, \sigma_{2,j}$ are the stress in the j direction of Members 1 and 2, respectively.

By the definition of Young's modulus under elastic deformation:

$$\sigma_{1,j} = E_{1,j}\epsilon_j \tag{6}$$

$$\sigma_{2,j} = E_{2,j}\epsilon_j \tag{7}$$

The total force can be written in terms of both the individual members' materials properties and the effective, composited, smeared:

$$F_j = (E_{1,j}A_1 + E_{2,j}A_2)\epsilon_j \tag{8}$$

$$F_j = E_{3,j}A_3\epsilon_j \tag{9}$$

Equating these two expressions:

$$E_{3,j} = \frac{E_{1,j}A_1 + E_{2,j}A_2}{A_1 + A_2} \tag{10}$$

or more generally

$$E_{j,par} = \frac{\sum E_j A}{\sum A} = \langle E_j \rangle_A \tag{11}$$

where $E_{j,par}$ is the smeared Young's modulus in the j direction of any number of parallel-composited ("*par*") members, and the sum is over members.

5.2 Axial-transverse Poisson's ratio of two members carrying an axial force in parallel

For the geometry of this case see Figure 1 (a). Take $\nu_{1,j,\perp}$ to be the j-transverse Poisson's ratio of Member 1, and $\nu_{2,j,\perp}$ to be the j-transverse Poisson's ratio of Member 2.

Consider $\nu_{3,j,\perp}$ to be the effective j-transverse Poisson's ratio that a member of cross-sectional area $A_3 = A_1 + A_2$ would have to have, in order to produce the same *transverse areal* strain ϵ_3, A due to Poisson effects under the same j-strain, ϵ_j. Transverse areal strain is the relative change in the transverse area, $\epsilon_A = A_{after}/A_{before} - 1$. Under the assumption of transverse-isotropy:

$$\epsilon_A = \epsilon_i + \epsilon_k = 2\epsilon_i = 2\epsilon_k = 2\epsilon_\perp \tag{12}$$

Plugging the definition of A_3 into the definition of ϵ_A above, we find

$$\epsilon_{3,A} = \frac{\epsilon_{1,A}A_1 + \epsilon_{2,A}A_2}{A_1 + A_2} \tag{13}$$

Recalling that $\epsilon_{1,A} = 2\epsilon_{1,\perp}$ and $\epsilon_{1,\perp} = \nu_{1,j,\perp}\epsilon_j$ (and likewise for Member 2:

$$\epsilon_{3,\perp} = \frac{\nu_{1,j,\perp}A_1 + \nu_{2,j,\perp}A_2}{A_1 + A_2}\epsilon_j \tag{14}$$

This is clearly the effective smeared composite Poisson's ratio:

$$\nu_{3,j,\perp} = \frac{\nu_{1,j,\perp}A_1 + \nu_{2,j,\perp}A_2}{A_1 + A_2} \tag{15}$$

or more generally

$$\nu_{j,\perp,par} = \frac{\sum \nu_{j,\perp}A}{\sum A} = \langle \nu_{j,\perp} \rangle_A \tag{16}$$

where $\nu_{j,\perp,par}$ is the smeared $j,\perp$ Poisson's ratio of any number of parallel-composited ("*par*") members, and the sum is over members.

5.3 Axial Young's modulus of two members carrying an axial force in series

For the geometry of this case see Figure 1 (b). Take L_1 to be the length in the j direction of Member 1, and L_2 to be the length in the j direction of Member 2.

Consider $E_{3,j}$ to be the effective Young's modulus in the j direction that a member of length $L_3 = L_1 + L_2$ would have to have, in order to produce the same total strain $\epsilon_{3,j} = L_{3,after}/L_{3,before} - 1$ under the same uniform stress σ_j.

By the definition of L_3 above,

$$(L_1 + L_2)\epsilon_{3,j} = \epsilon_{1,j}L_1 + \epsilon_{2,j}L_2 \tag{17}$$

Plugging in the definition of Young's modulus $\sigma_j = E_j\epsilon_j$ and rearranging

$$\epsilon_{3,j} = \frac{L_1/E_{1,j} + L_2/E_{2,j}}{L_1 + L_2}\sigma_j \tag{18}$$

The coefficient of σ_j on the right is the inverse Young's modulus:

$$E_{3,j} = \sigma_j/\epsilon_{3,j} = \frac{L_1 + L_2}{L_1/E_{1,j} + L_2/E_{2,j}} \tag{19}$$

or more generally

$$E_{j,ser} = \frac{\sum L}{\sum L/E_j} = \langle E_j^{-1} \rangle_l^{-1} \tag{20}$$

where $E_{j,ser}$ is the smeared Young's modulus in the j direction of any number of series-composited ("*ser*") members, and the sum is over members.

5.4 Axial-transverse Poisson's ratio of two members carrying an axial force in series

For the geometry of this case see Figure 1 (b).

Consider $\nu_{3,j,\perp}$ to be the effective j-transverse Poisson's ratio that a member of length $L_3 = L_1 + L_2$ would have to have, in order to produce the same *transverse volume* strain ϵ_3, V due to Poisson effects under the same uniform j-stress, σ_j. Transverse volume strain is the relative change in the volume, $\epsilon_V = V_{after}/V_{before} - 1$, neglecting axial j elongation/contraction. Under the assumption of transverse-isotropy:

$$\epsilon_V = \epsilon_i + \epsilon_k = 2\epsilon_i = 2\epsilon_k = 2\epsilon_\perp \tag{21}$$

The reason we have chosen volume V and not area A is that the cross-sectional area now changes under σ_j. The volume strain composites as the weighted average of the length because $V = \sum L \times A$:

$$\epsilon_{3,V} L_3 = \epsilon_{1,V} L_1 + \epsilon_{2,V} L_2 \tag{22}$$

Bringing σ_j and E_j into it:

$$\epsilon_{1,\perp} = \nu_{1,j,\perp} \epsilon_{1,j} = \nu_{1,j,\perp} \frac{\sigma_j}{E_{1,j}} \tag{23}$$

and likewise for Member 2.

Combining these equations:

$$\epsilon_{3,\perp} = \frac{\nu_{1,j,\perp} \frac{L_1}{E_{1,j}} + \nu_{2,j,\perp} \frac{L_2}{E_{2,j}}}{L_1 + L_2} \sigma_j \tag{24}$$

Recall in the last section we found $E_{3,j}$ which relates $\sigma_j = E_{3,j}\epsilon_{3,j}$:

$$\epsilon_{3,\perp} = \frac{\nu_{1,j,\perp} \frac{L_1}{E_{1,j}} + \nu_{2,j,\perp} \frac{L_2}{E_{2,j}}}{\frac{L_3}{E_3}} \epsilon_{3,j} \tag{25}$$

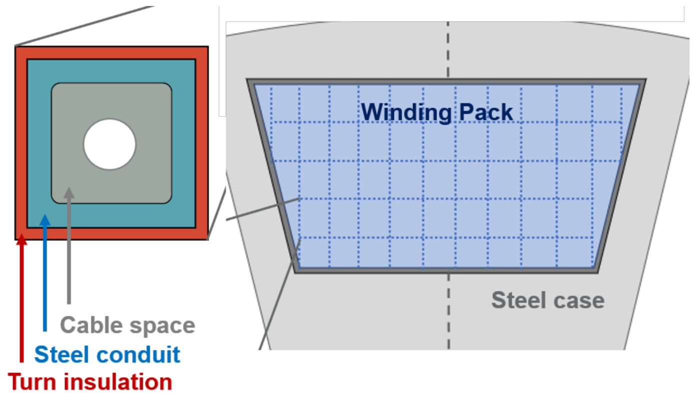

Figure 4: Illustration from the PROCESS TF coil documentation[5] showing the only available cable layout in the PROCESS code. Only the insulation and conduit currently contribute to vertical stiffness. Only the vertical slices which contain mostly steel currently contribute to the radial/toroidal stiffness. The compositing formulae are designed to allow more complete treatment of these, and ease the addition of new cable types.

The coefficient of $\epsilon_{3,j}$ is clearly the Poisson's ratio. With some rearranging we get:

$$\nu_{3,j,\perp} = \frac{\nu_{1,j,\perp}\frac{L_1}{E_{1,j}} + \nu_{2,j,\perp}\frac{L_2}{E_{2,j}}}{\frac{L_1}{E_{1,j}} + \frac{L_2}{E_{2,j}}} \tag{26}$$

or more generally

$$\nu_{j,\perp,ser} = \frac{\sum \nu_{j,\perp} L/E_j}{\sum L/E_j} = \langle \nu_{j,\perp} \rangle_{L/E_j} \tag{27}$$

where $\nu_{j,\perp,ser}$ is the smeared $j,\perp$ Poisson's ratio of any number of series-composited ("*ser*") members, and the sum is over members.

6 Application of these formulae to the PROCESS TF coil model

Many composite cable layouts may be considered by mixing and matching the compositing formulae. The PROCESS TF coil winding pack is assumed to be constructed from the vertically $\hat{z}$ aligned unit cells depicted in Figure 4.[5]

6.1 Vertical properties, $E_z, \nu_{z,\perp}$

A Fortran subroutine `eyngparallel` has been added to PROCESS file `sctfcoil.f90` in git branch `1205-tf-cond-stiffness`, which will presumably at some point be merged into the main `develop` branch. It takes three triplets of properties as arguments:

```
subroutine eyngparallel(eyoung_j_1, a_1, poisson_j_perp_1, & ! Inputs
                        eyoung_j_2, a_2, poisson_j_perp_2, & ! Inputs
                        eyoung_j_3, a_3, poisson_j_perp_3)   ! Outputs
```

The declaration of the third triplet `eyoung_j_3, a_3, poisson_j_perp_3` includes the `intent(in out)` option, so many members may be successively composited in parallel by passing the same triplet as triplets 2 and 3 of the arguments:

```
call eyngparallel(triplet1, triplet2, tripletOUT)
call eyngparallel(triplet3, tripletOUT, tripletOUT)
call eyngparallel(triplet4, tripletOUT, tripletOUT)
... etc.
```

So that `tripletOUT` would eventually have the smeared properties of all parallel-composited members.

To reproduce the existing PROCESS vertical-property results, only one call of `eyngparallel` would be required, as only the steel conduit and insulation contribute to the vertical stiffness. The successive calling of `eyngparallel` above allows the easy addition of the cable stiffness, and any other components that may some day be desired to include in the model.

6.2 Radial/toroidal properties, $E_\perp, \nu_\perp$

A Fortran subroutine `eyngseries` has been added to PROCESS file `sctfcoil.f90` in git branch `1205-tf-cond-stiffness`, which will presumably at some point be merged into the main `develop` branch. It takes three triplets of properties as arguments:

```
subroutine eyngseries(eyoung_j_1, a_1, poisson_j_perp_1, & ! Inputs
                      eyoung_j_2, a_2, poisson_j_perp_2, & ! Inputs
                      eyoung_j_3, a_3, poisson_j_perp_3)   ! Outputs
```

The handling of the three triplets of arguments is identical to `eyngparallel` above, so that the same successive call can be used to build up serially composited members of any number of members.

To reproduce the existing PROCESS transverse-property results, only one call of `eyngseries` would be required, as only the vertical slices[5] that are made up of mostly conduit steel are considered in the transverse stiffness computation. The utility of successively calling `eyngparallel` and `eyngseries` can now be seen, as it allows the addition of the other vertical slices in the following manner:

```
! Outermost legs, insulation
eyoung_p     = eyoung_ins
poisson_p    = poisson_ins
a_working    = 2*thicndut ! Thickness of the insulation

! Next inner legs, insulation and steel
call eyngseries(eyoung_steel, t_cable_oh + 2*t_cond_oh, poisson_steel, &
                eyoung_ins, 2*thicndut, poisson_ins, &
                eyoung_working, l_working, poisson_working)
! Add this to the properties we're accumulating
call eyngparallel(eyoung_working, 2*t_cond_oh, poisson_working, &
                  eyoung_p, a_working, poisson_p, &
                  eyoung_p, a_working, poisson_p)

! Next inner leg, the cable space
! Add insulation and steel
call eyngseries(eyoung_steel, 2*t_cond_oh, poisson_steel, &
eyoung_ins, 2*thicndut, poisson_ins, &
eyoung_working, l_working, poisson_working)
! Add cable
call eyngseries(eyoung_cab, t_cable_oh, poisson_cab, &
                eyoung_working, l_working, poisson_working, &
                eyoung_working, l_working, poisson_working)
! Add this to the properties we're accumulating
call eyngparallel(eyoung_working, t_cable_oh, poisson_working, &
                  eyoung_p, a_working, poisson_p, &
                  eyoung_p, a_working, poisson_p)
```

In this manner (if there are no typos), `eyoung_p` and `poisson_p` start out considering only the outermost two vertical slices, which are entirely insulation. Then, `eyoung_working` and `poisson_working` are set to the properties of the next-inner vertical slices, which is a serial-composite of a layer of insulation and a layer of steel. This serial composite is then parallel composited back into `eyoung_p` and `poisson_p`. Finally, a multi-step `eyngseries` call serial-composites first the insulation and steel, then the cable space, of the innermost vertical slice, into `eyoung_working` and `poisson_working`. A final `eyngparallel` parallel-composites these properties back into `eyoung_p` and `poisson_p`, and we are done.

7 References

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