Undulation matrix calculations for 2D Hydrogen beam emission spectroscopy absolute electron density fluctuation reconstruction

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I. INTRODUCTION

The following calculations show the steps and results of the undulation matrix minimization method. These calculations supplement the ones in the paper titled "Absolute electron density fluctuation reconstruction for two-dimensional Hydrogen beam emission spectroscopy", however, due to their length they are not part of the paper or its appendix.

The following notations are used thereafter:

- P_{ij}: Parameter matrix, absolute electron density at the r_i and z_j locations.
- A_{ij}: Two dimensional basis function for series decomposition.
- $H(r, z)$: Undulation function at r, z points.
- $n_e(r, z), n_e^0(r, z), \tilde{n}_e(r, z, t)$: Electron density, electron density profile, and electron density fluctuations at r, z points, respectively.
- m: number of measurement points in radial direction.
- n: number of measurement points in poloidal direction.
- single underline: denotes vectors.
- double underline: denotes matrices.

If electron density fluctuations in the plasma are below 10%, it can be assumed that the measured light fluctuations are linearly dependent on the electron density fluctuations. Thus the perturbations are in the first order and the higher order terms can be neglected. The light intensity distribution $S(r, z, t)$ can be written as a function of the electron density distribution $n_e(r, z, t)$ as

$$S(r, z, t) = \mathcal{F}\{n_e(r', z', t)\} \qquad (1)$$
$$0 \leq z' \leq z$$

.

The electron density can be written as a sum of time independent and a fluctuating time dependent term as

$$n_e(r,z,t) = n_e^0(r,z) + \tilde{n}_e(r,z,t). \tag{2}$$

If the perturbations are small relative to the average electron density for the entire measurement range, one can linearize the transformation $\mathcal{F}$ in equation 1 as

$$\begin{aligned} S(r,z.t) &= S^0(r,z) + \tilde{S}(r,z,t) \\ S^0(r,z) &= \mathcal{F}\{n_e^0(r,z)\} \\ \tilde{S}(r,z,t) &\approx \int_0^z \int_0^r \tilde{n}_e(r',z',t) h(r,z,r',z') dr' dz' \end{aligned} \tag{3}$$

where $h(r,z,r',z')$ is the linear transfer function describing the effect of a small density fluctuation at (r',z') on the light signal at (r,z).

II. DERIVING THE UNDULATION MATRIX H_{ijkl}

The light is measured at discreet spatial positions, therefore, the problem needs to be translated to discreet from continuous. We define the following series expansion as

$$\tilde{n}_e(r,z,t) = \sum_{i=1}^{n_r} \sum_{j=1}^{n_z} \tilde{P}_{ij}(t) A_{ij}(r,z) \tag{4}$$

where $\tilde{P}_{ij}(t)$ are the densities at (r_i, z_j) and $A_{ij}(r,z)$ are the basis functions of the series expansion. The latter are described in the following section.

In order to translate the problem to discreet, the transfer function needs to be discretized, as well, as

$$\tilde{S}(r,z,t) \approx \int_0^z \int_0^r \sum_{i=1}^{n_r} \sum_{j=1}^{n_z} \tilde{P}_{ij}(t) A_{ij}(r',z') h(r,z,r',z') dr' dz' \tag{5}$$

$$\tilde{S}(r,z,t) \approx \sum_{i=1}^{n_r} \sum_{j=1}^{n_z} \int_0^z \int_0^r A_{ij}(r',z') h(r,z,r',z') dr' dz' \tilde{P}_{ij}(t) \tag{6}$$

$$\tilde{S}(r_i^S, z_j^S, t) = \tilde{S}_{ij}(t) \approx \sum_{k=1}^{n_r} \sum_{l=1}^{n_z} M_{ijkl} \tilde{P}_{kl}(t) \tag{7}$$

$$M_{ijkl} = \int_0^{z_j^S} \int_0^{r_i^S} A_{ij}(r', z') h(r_k^S, z_j^S, r', z') dr' dz' \tag{8}$$

We introduce the $H(t)$ undulation, which quantifies the waviness of the electron density fluctuations.

$$H(t) = \int_A \left(\frac{\partial \tilde{n}_e(r, z, t)}{\partial r} \right)^2 + \left(\frac{\partial \tilde{n}_e(r, z, t)}{\partial z} \right)^2 drdz \tag{9}$$

This function needs to be discretized, as well, which is done by substituting the series expanded form of $\tilde{n}_e(r, z, t)$ into H(t).

$$H(t) = \int_A \left(\sum_{i=1}^{n_r} \sum_{j=1}^{n_z} \tilde{P}_{ij}(t) \frac{\partial A_{ij}(r, z)}{\partial r} \right)^2 + \left(\sum_{i=1}^{n_r} \sum_{j=1}^{n_z} \tilde{P}_{ij}(t) \frac{\partial A_{ij}(r, z)}{\partial z} \right)^2 \tag{10}$$

$$H(t) = \int_A \sum_{ijkl} \left(\tilde{P}_{ij}(t) \tilde{P}_{kl}(t) \frac{\partial A_{ij}(r, z)}{\partial z} \frac{\partial A_{kl}(r, z)}{\partial z} + \tilde{P}_{ij}(t) \tilde{P}_{kl}(t) \frac{\partial A_{ij}(r, z)}{\partial r} \frac{\partial A_{kl}(r, z)}{\partial r} \right) drdz \tag{11}$$

$$H(t) = \sum_{ijkl} \tilde{P}_{ij}(t) \tilde{P}_{kl}(t) \int_A \left(\frac{\partial A_{ij}(r, z)}{\partial r} \frac{\partial A_{kl}(r, z)}{\partial r} + \frac{\partial A_{ij}(r, z)}{\partial z} \frac{\partial A_{kl}(r, z)}{\partial z} \right) drdz \tag{12}$$

$$H(t) = \sum_{ijkl} \tilde{P}_{ij}(t) \tilde{P}_{kl}(t) H_{ijkl} \tag{13}$$

where H_{ijkl} is the undulation matrix and it is defined by

$$H_{ijkl} = \int_A \left(\frac{\partial A_{ij}(r, z)}{\partial r} \frac{\partial A_{kl}(r, z)}{\partial r} + \frac{\partial A_{ij}(r, z)}{\partial z} \frac{\partial A_{kl}(r, z)}{\partial z} \right) drdz \tag{14}$$

.

III. BASIS FUNCTION $A_{ij}(r, z)$

The basis functions perform interpolation for the r and z continuous directions in essence. These functions can be chosen as any polynomial functions as long as they perform interpolation on the two-dimensional plane. At the location of the nearest neighbor its value equals 1 while at the other neighbors its value is zero. Otherwise it is the chosen polynomial. For the undulation minimization, due to the relatively low number of detectors in beam emission spectroscopy measurements, bi-linear basis functions were chosen. These basis functions can be written as

$$A_{ij}(r,z) = 0 \begin{cases} r \leq r_{i-1} \\ r > r_{i+1} \\ z \leq z_{i-1} \\ z > z_{i+1} \end{cases} \tag{15}$$

$$A_{ij}(r,z) = \frac{r - r_{i-1}}{r_i - r_{i-1}} \frac{z - z_{j-1}}{z_j - z_{j-1}} \begin{cases} r_{i-1} < r \leq r_i \\ z_{j-1} < z \leq z_j \end{cases} \tag{16}$$

$$A_{ij}(r,z) = \frac{r - r_{i+1}}{r_i - r_{i+1}} \frac{z - z_{j-1}}{z_j - z_{j-1}} \begin{cases} r_i < r \leq r_{i+1} \\ z_{j-1} < z \leq z_j \end{cases} \tag{17}$$

$$A_{ij}(r,z) = \frac{r - r_{i-1}}{r_i - r_{i-1}} \frac{z - z_{j+1}}{z_j - z_{j+1}} \begin{cases} r_{i-1} < r \leq r_i \\ z_j < z \leq z_{j+1} \end{cases} \tag{18}$$

$$A_{ij}(r,z) = \frac{r - r_{i+1}}{r_i - r_{i+1}} \frac{z - z_{j+1}}{z_j - z_{j+1}} \begin{cases} r_i < r \leq r_{i+1} \\ z_j < z \leq z_{j+1} \end{cases} \tag{19}$$

.

To calculate the undulation matrix, one needs to write down the derivatives of the basis functions. Due to their bi-linear nature, they can be written in the following way to simplify the calculations:

$$A_{ij}(r,z) = a_i(r) b_j(z)$$

$$a_i = \frac{r - r_{i\pm1}}{r_i - r_{i\pm1}}$$

$$b_j = \frac{z - z_{j\pm1}}{z_j - z_{j\pm1}} \tag{20}$$

$$\frac{\partial A_{ij}}{\partial r} = \frac{\partial a_i}{\partial r} b_j$$

$$\frac{\partial A_{ij}}{\partial z} = a_i \frac{\partial b_j}{\partial z} \tag{21}$$

$$\frac{\partial a_i}{\partial r} = \frac{1}{r_i - r_{i\pm 1}}$$

$$\frac{\partial b_j}{\partial z} = \frac{1}{z_j - z_{j\pm 1}} \tag{22}$$

IV. DERIVING THE SIMPLIFIED UNDULATION MATRIX

The undulation matrix can then be written as

$$H_{ijkl} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left(b_j b_l \frac{\partial a_i}{\partial r} \frac{\partial a_k}{\partial r} + a_i a_k \frac{\partial b_j}{\partial z} \frac{\partial b_l}{\partial z} \right) dr dz \tag{23}$$

a_i and b_j are only functions of r and z, respectively, the integral can be written as a multiplication of two integrals as

$$H_{ijkl} = \int_{-\infty}^{+\infty} b_j b_l dz \int_{-\infty}^{+\infty} \frac{\partial a_i}{\partial r} \frac{\partial a_k}{\partial r} dr + \int_{-\infty}^{+\infty} a_i a_k dr \int_{-\infty}^{+\infty} \frac{\partial b_j}{\partial z} \frac{\partial b_l}{\partial z} dz \tag{24}$$

.

The individual integrals can be calculated as

$$\int a_i a_k dz = \int \frac{r - r_{i\pm 1}}{r_i - r_{i\pm 1}} \frac{r - r_{k\pm 1}}{r_k - r_{k\pm 1}} dz =$$
$$= \frac{2r^3 - 3r^2 r_{k\pm 1} - 3r^2 r_{i\pm 1} + r r_{i\pm 1} r_{k\pm 1}}{(r_i - r_{i\pm 1})(r_k - r_{k\pm 1})} \tag{25}$$

$$\int b_j b_l dz = \int \frac{z - z_{j\pm 1}}{z_j - z_{j\pm 1}} \frac{z - z_{l\pm 1}}{z_l - z_{l\pm 1}} dz =$$
$$= \frac{2z^3 - 3z^2 z_{l\pm 1} - 3z^2 z_{j\pm 1} + z z_{j\pm 1} z_{l\pm 1}}{(z_j - z_{j\pm 1})(z_l - z_{l\pm 1})} \tag{26}$$

$$\int \frac{\partial a_i}{\partial r} \frac{\partial a_k}{\partial r} dr = \frac{1}{r_i - r_{i\pm 1}} \frac{1}{r_k - r_{k\pm 1}} r \tag{27}$$

$$\int \frac{\partial b_j}{\partial z}\frac{\partial b_l}{\partial z}dz = \frac{1}{z_j - z_{j\pm1}}\frac{1}{z_l - z_{l\pm1}}z \tag{28}$$

.

Then one can rewrite the undulation matrix as

$$H_{ijkl} = H^a_{ijkl} + H^b_{ijkl}$$

$$H^a_{ijkl} = \int_Z b_j b_l dz \int_R \frac{\partial a_i}{\partial r}\frac{\partial a_k}{\partial r}dr =$$

$$= \left.\frac{2z^3 - 3z^2 z_{l\pm1} - 3z^2 z_{j\pm1} + z z_{j\pm1} z_{l\pm1}}{(z_j - z_{j\pm1})(z_l - z_{l\pm1})}\right|_Z \cdot \left.\frac{1}{r_i - r_{i\pm1}}\frac{1}{r_k - r_{k\pm1}}r\right|_R$$

$$H^b_{ijkl} = \int_R a_i a_k dr \int_Z \frac{\partial b_j}{\partial z}\frac{\partial b_l}{\partial z}dz =$$

$$= \left.\frac{2r^3 - 3r^2 r_{k\pm1} - 3r^2 r_{i\pm1} + r r_{i\pm1} r_{k\pm1}}{(r_i - r_{i\pm1})(r_k - r_{k\pm1})}\right|_R \cdot \left.\frac{1}{z_j - z_{j\pm1}}\frac{1}{z_l - z_{l\pm1}}z\right|_Z \tag{29}$$

.

The integrals have finite values if the basis functions overlap, i.e., when $k = i$, $k = i \pm 1$, $j = l$ or $j = l \pm 1$. Therefore, only nine elements will have finite values in the matrix, the others are zero. The non zero elements are

- 1) $k = i - 1$ and $l = j - 1$: $H_{iji-1j-1}$
- 2) $k = i - 1$ and $l = j$: H_{iji-1j}
- 3) $k = i - 1$ and $l = j + 1$: $H_{iji-1j+1}$
- 4) $k = i$ and $l = j - 1$: H_{ijij-1}
- 5) $k = i$ and $l = j$: H_{ijij}
- 6) $k = i$ and $l = j + 1$: H_{ijij+1}
- 7) $k = i + 1$ and $l = j - 1$: $H_{iji+1j-1}$
- 8) $k = i + 1$ and $l = j$: H_{iji+1j}

- 9) $k = i + 1$ and $l = j + 1$: $H_{iji+1j+1}$

.

V. CALCULATION OF THE INTEGRALS

The integrals can be written as a sum of two terms. That is denoted with a) and b) as

$$
\begin{aligned}
a) \int_Z b_j b_l dz \int_R \frac{\partial a_i}{\partial r}\frac{\partial a_k}{\partial r} dr \\
b) \int_R a_i a_k dr \int_Z \frac{\partial b_j}{\partial z}\frac{\partial b_l}{\partial z} dz
\end{aligned}
\tag{30}
$$

.

The plus and minus signs in the indices need to be treated carefully, e.g., if one is calculating the integral from r_{i-1} to r_i for 3) then the minus sign is taken for i, but the plus sign is taken for k because the overlap of the basis function a_k is on its positive side.

5) $k = i$ **and** $l = j$

For the first case, the whole calculation is given and for the rest, only the results are shown.

$$H_{ijij} = H^a_{ijij} + H^b_{ijij} \tag{31}$$

$$H^a_{ijij} = \int_Z b_j b_j dz \int_R \frac{\partial a_i}{\partial r}\frac{\partial a_i}{\partial r} dr =$$

$$= \left[\int_{z_{j-1}}^{z_j} b_j b_j dz + \int_{z_j}^{z_{j+1}} b_j b_j dz\right] \cdot \left[\int_{r_{i-1}}^{r_i} \frac{\partial a_i}{\partial r}\frac{\partial a_i}{\partial r} dr + \int_{r_i}^{r_{i+1}} \frac{\partial a_i}{\partial r}\frac{\partial a_i}{\partial r} dr\right] \tag{32}$$

$$H^b_{ijij} = \int_R a_i a_i dz \int_Z \frac{\partial b_j}{\partial z}\frac{\partial b_j}{\partial z} dr =$$

$$= \left[\int_{r_{i-1}}^{r_i} a_i a_i dz + \int_{z_j}^{z_{j+1}} b_j b_j dz\right] \cdot \left[\int_{r_{i-1}}^{r_i} \frac{\partial a_i}{\partial r}\frac{\partial a_i}{\partial r} dr + \int_{r_i}^{r_{i+1}} \frac{\partial a_i}{\partial r}\frac{\partial a_i}{\partial r} dr\right] \tag{33}$$

In the lower intervals, e.g., between r_{i-1} and r_i the negative is taken from the plus-minus sign, while for the upper intervals the positive sign is taken. For the overlaps, see Fig. 5 in the publications.

$$\begin{aligned} H^a_{ijij} = &\left[\frac{2z_j^3 - 3z_j^2 z_{j-1} - 3z_j^2 z_{j-1} + 6z_j z_{j-1}^2}{6(z_j - z_{j-1})^2} - \frac{2z_{j-1}^3 - 3z_{j-1}^2 z_{j-1} - 3z_{j-1}^2 z_{j-1} + 6z_{j-1} z_{j-1}^2}{6(z_j - z_{j-1})^2} + \right.\\ &\left. + \frac{2z_{j+1}^3 - 3z_{j+1}^2 z_{j+1} - 3z_{j+1}^2 z_{j+1} + 6z_{j+1} z_{j+1}^2}{6(z_j - z_{j+1})^2} - \frac{2z_j^3 - 3z_j^2 z_{j+1} - 3z_j^2 z_{j+1} + 6z_j z_{j+1}^2}{6(z_j - z_{j+1})^2}\right] \cdot \\ &\cdot \left[\frac{1}{(r_i - r_{i-1})^2} r_i - \frac{1}{(r_i - r_{i-1})^2} r_{i-1} + \frac{1}{(r_i - r_{i+1})^2} r_{i+1} - \frac{1}{(r_i - r_{i+1})^2} r_i\right] \end{aligned} \tag{34}$$

$$\begin{aligned} H^b_{ijij} = &\left[\frac{2r_i^3 - 3r_i^2 r_{i-1} - 3r_i^2 r_{i-1} + 6r_i r_{i-1}^2}{6(r_i - r_{i-1})^2} - \frac{2r_{i-1}^3 - 3r_{i-1}^2 r_{i-1} - 3r_{i-1}^2 r_{i-1} + 6r_{i-1} r_{i-1}^2}{6(r_i - r_{i-1})^2} + \right.\\ &\left. + \frac{2r_{i+1}^3 - 3r_{i+1}^2 r_{i+1} - 3r_{i+1}^2 r_{i+1} + 6r_{i+1} r_{i+1}^2}{6(r_i - r_{i+1})^2} - \frac{2r_i^3 - 3r_i^2 r_{i+1} - 3r_i^2 r_{i+1} + 6r_i r_{i+1}^2}{6(r_i - r_{i+1})^2}\right] \cdot \\ &\cdot \left[\frac{1}{(z_j - z_{j-1})^2} z_j - \frac{1}{(z_j - z_{j-1})^2} z_{j-1} + \frac{1}{(z_j - z_{j+1})^2} z_{j+1} - \frac{1}{(z_j - z_{j+1})^2} z_j\right] \end{aligned} \tag{35}$$

After simplifying and adding the two terms we arrive at the following expression for the matrix element

$$H_{ijij} = \frac{z_{j+1} - z_{j-1}}{3} \cdot \left[\frac{1}{r_i - r_{i-1}} + \frac{1}{r_{i+1} - r_i} \right] + \frac{r_{i+1} - r_{i-1}}{3} \cdot \left[\frac{1}{z_j - z_{j-1}} + \frac{1}{z_{j+1} - z_j} \right] \quad (36)$$

.

1) $k = i - 1$ **and** $l = j - 1$

$$H_{iji-1j-1} = \frac{z_j - z_{j-1}}{6} \cdot \frac{1}{r_{i-1} - r_i} + \frac{r_i - r_{i-1}}{6} \cdot \frac{1}{z_{j-1} - z_j} \quad (37)$$

2) $k = i - 1$ **and** $l = j$

$$H_{iji-1j} = \frac{z_{j+1} - z_{j-1}}{3} \cdot \frac{1}{r_{i-1} - r_i} + \frac{r_i - r_{i-1}}{6} \cdot \left[\frac{1}{z_j - z_{j-1}} + \frac{1}{z_{j+1} - z_j} \right] \quad (38)$$

3) $k = i - 1$ **and** $l = j + 1$

$$H_{iji-1j+1} = \frac{z_{j+1} - z_j}{6} \cdot \frac{1}{r_{i-1} - r_i} + \frac{r_i - r_{i-1}}{6} \cdot \frac{1}{z_j - z_{j+1}} \quad (39)$$

4) $k = i$ **and** $l = j - 1$

$$H_{ijij-1} = \frac{z_j - z_{j-1}}{6} \cdot \left[\frac{1}{r_i - r_{i-1}} + \frac{1}{r_{i+1} - r_i} \right] + \frac{r_{i+1} - r_{i-1}}{3} \cdot \frac{1}{z_{j-1} - z_j} \quad (40)$$

6) $k = i$ **and** $l = j + 1$

$$H_{ijij+1} = \frac{z_{j+1} - z_j}{6} \cdot \left[\frac{1}{r_i - r_{i-1}} + \frac{1}{r_{i+1} - r_i} \right] + \frac{r_{i+1} - r_{i-1}}{3} \cdot \frac{1}{z_j - z_{j+1}} \quad (41)$$

7) $k = i+1$ **and** $l = j-1$

$$H_{iji+1j-1} = \frac{z_j - z_{j-1}}{6} \cdot \frac{1}{r_i - r_{i+1}} + \frac{r_{i+1} - r_i}{6} \cdot \frac{1}{z_{j-1} - z_j} \tag{42}$$

8) $k = i+1$ **and** $l = j$

$$H_{iji+1j} = \frac{z_{j+1} - z_{j-1}}{3} \cdot \frac{1}{r_i - r_{i+1}} + \frac{r_{i+1} - r_i}{6} \cdot \left[\frac{1}{z_j - z_{j-1}} + \frac{1}{z_{j+1} - z_j}\right] \tag{43}$$

9) $k = i+1$ **and** $l = j+1$

$$H_{iji+1j+1} = \frac{z_{j+1} - z_j}{6} \cdot \frac{1}{r_i - r_{i+1}} + \frac{r_{i+1} - r_i}{6} \cdot \frac{1}{z_j - z_{j+1}} \tag{44}$$

VI. UNDULATION MINIMIZATION

After calculating the undulation matrix based on the bi-linear basis functions, the minimum of the χ function needs to be calculated which is defined by the following expression:

$$\chi(\underline{\underline{\tilde{P}}}) = \sum_{i=1}^{n_r}\sum_{j=1}^{n_z}\sum_{k=1}^{n_r}\sum_{l=1}^{n_z} H_{ijkl}\tilde{P}_{ij}\tilde{P}_{kl} + K\sum_{m=1}^{m_r}\sum_{n=1}^{m_z}\left(\tilde{S}_{mn} - \sum_{k=1}^{n_r}\sum_{l=1}^{n_z} M_{mnkl}\tilde{P}_{kl}\right)^2 \sigma_{mn}^{-2} \tag{45}$$

.

The minimum of the function needs to be found with respect to $\underline{\underline{P}}$, thus the χ function needs to be partially differentiated with P_{mn}. Before that, one should expand the upper expression to the following.

$$\chi(\underline{\underline{\tilde{P}}}) = \sum_{ijkl=1}^{n_r n_z n_r n_z} H_{ijkl}\tilde{P}_{ij}\tilde{P}_{kl} + K\sum_{m,n=1}^{m_r m_z}\Bigg(\tilde{S}_{mn}^2 - 2\tilde{S}_{mn}\sum_{kl=1}^{n_r n_z} M_{mnkl}\tilde{P}_{kl} + \\ + \sum_{kl=1}^{n_r n_z}\sum_{k',l'=1}^{n_r n_z} M_{mnkl}\tilde{P}_{kl}M_{mnk'l'}\tilde{P}_{k'l'}\Bigg)\sigma_{mn}^{-2} \tag{46}$$

Calculating the differential:

$$\frac{\partial \chi(\underline{\underline{\tilde{P}}})}{\partial \tilde{P}_{op}} = 2\sum_{ij} H_{ijop}\tilde{P}_{ij} - 2K\sum_{mn} M_{mnop}\sigma_{mn}^{-2}\tilde{S}_{mn} + 2K\sum_{mnkl} M_{mnkl}M_{mnop}\sigma_{mn}^{-2}\tilde{P}_{kl} = 0$$

$$K\sum_{mn} M_{mnop}\sigma_{mn}^{-2}\tilde{S}_{mn} = \sum_{kl}\left(H_{klop} + K\sum_{mn} M_{mnkl}M_{mnop}\sigma_{mn}^{-2}\right)\tilde{P}_{kl} \quad (47)$$

The latter equation can be rewritten to a matrix equation after reorganizing the hyper-matrix elements into two dimensional matrices as

$$\begin{aligned} m' &= n \cdot m_r + m \\ o' &= p \cdot n_r + o \\ k' &= l \cdot n_r + k \end{aligned} \quad (48)$$

Using this notation one can arrive at the following expression:

$$K\sum_{m'} M_{m'o'}\sigma_{m'}^{-2}\tilde{S}_{m'} = \sum_{k'}\left(H_{k'o'} + K\sum_{m'} M_{m'k'}M_{m'o'}\sigma_{m'}^{-2}\right)\tilde{P}_{k'} \quad (49)$$

The equation can be rewritten as a matrix equation, which can be solved by finding the inverse matrix of

$\underline{\underline{N}}$.

$$\begin{aligned} K\underline{\underline{M}}'^T\underline{\tilde{S}} &= \left(\underline{\underline{H}}^T + K\underline{\underline{M}}'^T\underline{\underline{M}}\right)\underline{\tilde{P}} \\ \underline{\tilde{P}} &= K\left(\underline{\underline{H}}^T + K\underline{\underline{M}}'^T\underline{\underline{M}}\right)^{-1}\underline{\underline{M}}'^T\underline{\tilde{S}} \\ M'_{m'o'} &= M_{m'o'}\sigma_{m'}^{-2} \end{aligned} \quad (50)$$

Since K is an arbitrary number, the solution is not optimal. The following condition is applied in order to find the solution, where the smoothing is optimal and its level equals the measurement uncertainty.

$$\chi(\underline{\underline{\tilde{P}}}) = (m_r m_z)^{-1}\sum_{m'}\left(\tilde{S}_{m'} - \sum_{k'} M_{m'k'}\tilde{P}_{k'}\right)^2\sigma_{m'}^{-2} = 1 \quad (51)$$

After rewriting this expression and the previous matrix equation into a single matrix equation, one can solve it by a traditional root finding method, e.g., bi-section.